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A GENERALIZATION OF THE REGULA FALSI FOR THE APPROXIMATE CALCULATION OF NULL POINTS OF A SYSTEM OF NONLINEAR EQUATIONS

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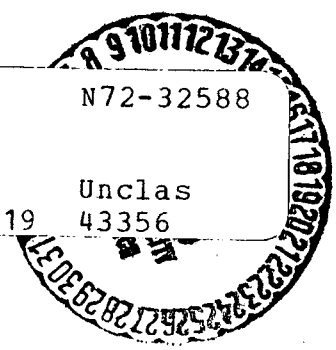

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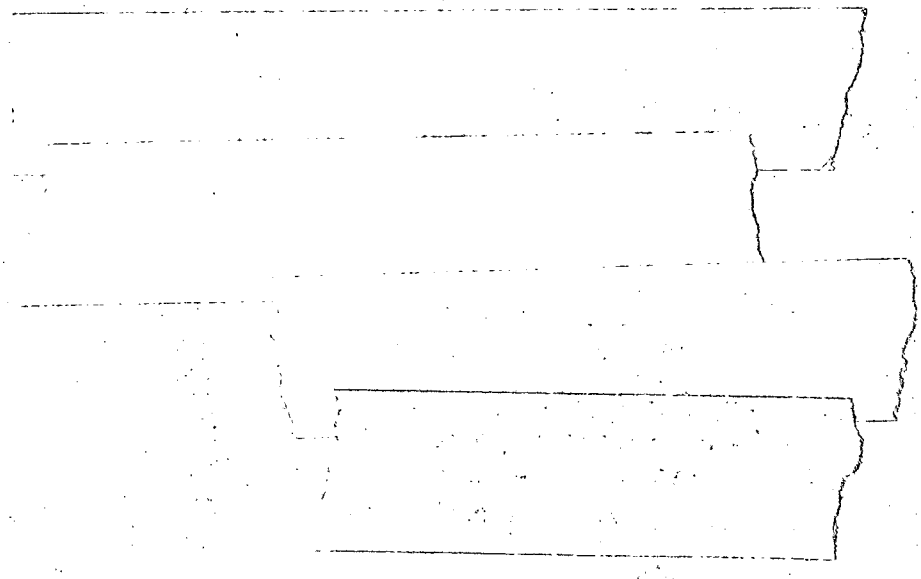
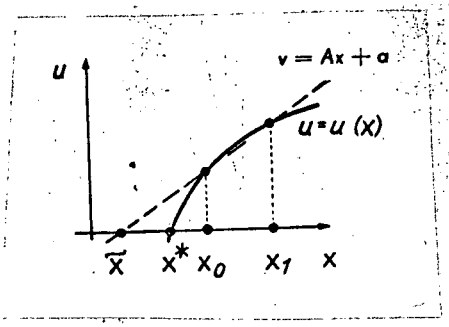


A GENERALIZATION OF THE REGULA FALSI FOR THE APPROXIMATE CALCULATION OF NULL POINTS OF A SYSTEM OF NONLINEAR EQUATIONS

By Leonhard Bittner

Let $x = (\xi_1, \xi_2 \dots \xi_n)'$ a vector of a real n-dimensional euclidean space R_n and $u(x) = [u_{(1)}(x), u_{(2)}(x), \dots u_{(n)}(x)]'$ be a function which is defined by R_n and projected into R_n.

It is required to determine a certain null point x^* of $u(x)$, the existence of which is here assumed.



It is now the task of the regula falsi (n = 1!) to substitute the nonlinear relationship between x and u by a linear relationship $v = Ax + a$ which coincides with the nonlinear relationship $u = u(x)$ in the two points $P_0 = (x_0, u_0)$ and $P_1 = (x_1, u_1)$ and to consider its null point $\tilde{x}$ as an approximation for x^*.

For the general case that ($n > 1$) we may obtain a linear substitute for a nonlinear vector function $u = u(x)$ by a related transformation of the form

$$v = Ax + a. \tag{1}$$

In this equation $v \in R_n$, $a \in R_n$ and A is a (n, n) - Matrix. Let us assume that for (n + 1) point

$$x_0, x_1, \ldots, x_n$$

the values of the function

$$u_0 = u(x_0),\ u_1 = u(x_1) \ldots u_n = u(x_n)$$

of a nonlinear function are known. In order to determine A and a we require that the nonlinear function and its linear substitution will give coinciding functional values in the points x_j ($j = 0 \ldots n$), so that

$$v_j = Ax_j + a = u_j \quad (j = 0, 1 \ldots n) \tag{2}$$

and consider the null point $\tilde{x}$ of the related transformation as a new approximation for the required value of x^*.

From equation 2 we see

$$u_j - u_n = A(x_j - x_n) \quad (j = 0, 1 \ldots n-1)$$

and

$$\begin{aligned} &\left(u_0 - u_n,\ u_1 - u_n \ldots u_{n-1} - u_n\right) \\ &= A\left(x_0 - x_n,\ x_1 - x_n \ldots x_{n-1} - x_n\right), \end{aligned} \tag{3}$$

on the other hand we have from equations (1) and (2)

$$v - u_n = A(x - x_n).$$

For v = 0 we obtain therefore

$$\begin{aligned} \bar{x} = x_{n+1} &= x_n - A^{-1} u_n \\ &= x_n - (x_0 - x_n,\ x_1 - x_n \ldots x_{n-1} - x_n)\,(u_0 - u_n,\ u_1 - u_n \ldots \\ &\qquad u_{n-1} - u_n)^{-1} u_n \\ &:= x_n - L_n \cdot u_n, \end{aligned} \tag{4}$$

This is assuming that the Matrix composed of the vector u_j is a non-singular one.

Equation (4) we may compare with a somewhat related relationship. If we assume that $x_k (k > n)$ are the calculated approximation and that x_{0k}, x_{1k}, ...x_{nk} are some previously obtained approximations for x* we obtain

$$x_{k+1} = x_k - \left(x_{0_k} - x_{n_k}, \ldots, x_{(n-1)_k} - x_{n_k}\right)\left(u_{0_k} - u_{n_k}, \ldots, u_{(n-1)_k} - u_{n_k}\right)^{-1} u(x_k)$$
$$:= x_k - L_k \cdot u_k. \qquad (4')$$

Remark: The rule expressed in equation (4) is treated by Professor Dr. H. Heinrich in his lectures for several years now with different motivation. It will be shown presently that under certain circumstances L_n is an approximation for the inverse functional Matrix for points approximately around x*. The original question posed by Professor Dr. H. Heinrich was: how may we substitute the functional Matrix by finite expressions if we do not wish to form the required derivatives or if these derivatives are numerically not available?

Our next task consists of finding the conditions under which the successive approximations calculated according to equation (4) or (4') form a convergent series. To begin with, let us make four obvious assumptions.

$\alpha)$ $u_{(i)}(x) = u_{(i)}(\xi_1 \ldots \xi_n)$ has two partial derivatives for the values (i = 1,2...n)

$$\beta) \quad \left|\frac{\partial^2 u_{(i)}(x)}{\partial \xi_k \, \partial \xi_l}\right| \leqq \varkappa \; (i, k, l = 1, 2 \ldots n) \text{ auf } K = \{x : |x - x^*| \leqq \varrho\}$$
$$\gamma) \quad x_0, x_1 \ldots x_n \in K$$
$$\delta) \quad \left|\det\left(\frac{\partial u_{(i)}}{\partial \xi_k}\right)\right|_{x = x^*} \neq 0. \qquad (5)$$

Two further conditions which determine the distance of $x_j (j = 0, 1...n)$ from x^* and the relative position of x_j or $x_{jk} (j = 0,...n)$ will soon be derived. For a more convenient preliminary approximation we use some simple rules for Frechet functions. Because of (5) $u(x) \in R_n$ is twice Frechet differentiable, and for the vectors $z \in R_n$, $w \in R_n$ we have

$$\left.\begin{aligned} u'(x)\,z &= \left(\frac{\partial u_{(i)}(x)}{\partial \xi_k}\right) \cdot z \in R_n \\ u''(x)\,zw &= \begin{bmatrix} z'\left(\frac{\partial^2 u_{(1)}}{\partial \xi_k\, \partial \xi_l}\right) w \\ \vdots \\ z'\left(\frac{\partial^2 u_{(n)}}{\partial \xi_k\, \partial \xi_l}\right) w \end{bmatrix} \in R_n. \end{aligned}\right\} \tag{6}$$

By applying the Schwarzch' inequality we obtain for the bilinear operator u''(x)

$$\begin{aligned} |u''(x)\,zw| &= \sqrt{\sum_{i=1}^{n} \left| y'\left(\frac{\partial^2 u_{(i)}}{\partial \xi_k\, \partial \xi_l}\right) w \right|^2} \\ &\leqq \sqrt{\sum_{i=1}^{n} \left\| \left(\frac{\partial^2 u_{(i)}}{\partial \xi_k\, \partial \xi_l}\right) \right\|^2} \cdot |z| \cdot |w|, \end{aligned}$$

where vectors in pythagorian-norm shown by single vertical lines and where double vertical lines around the Matrices and the F derivatives u'(x), u''(x) show their norms.

In case of $x \in K$, we obtain because of (5β)

$$\left\| \left(\frac{\partial^2 u_{(i)}(x)}{\partial \xi_k\, \partial \xi_l}\right) \right\|^2 \leqq \sum_{k,l=1}^{n} \left| \frac{\partial^2 u_{(i)}}{\partial \xi_k\, \partial \xi_l} \right|^2 \leqq n^2 x^2,$$

and

$$|u''(x)\,zw| \leqq \sqrt{n^3 \varkappa^2}\,|z| \cdot |w| = n\sqrt{n}\,\varkappa |z| \cdot |w|,$$

that means

$$\|u''(x)\| \leqq n\sqrt{n}\,\varkappa, \quad (x \in K). \tag{7}$$

The following inequalities are known to exist [1]

$$\left.\begin{aligned} |u(x) - u(y)| &\leqq \|u'(\bar{x})\| \cdot |x - y|, \\ \bar{x} &= \sigma x + (1-\sigma)\,y, \quad 0 < \sigma < 1 \\ \|u'(x) - u'(y)\| &\leqq \|u''(\bar{y})\| \cdot |x - y|, \\ \bar{y} &= \tau x + (1-\tau)\,y, \quad 0 < \tau < 1 \\ |u(x) - u(y) - u'(y)\,(x-y)| &\leqq \frac{1}{2}\|u''(\hat{x})\| \cdot |x-y|^2, \\ \hat{x} &= \vartheta x + (1-\vartheta)\,y, \quad 0 < \vartheta < 1. \end{aligned}\right\} \tag{8}$$

For the following we also have to know the norm of the inverse nonsingular Matrix which was derived from unit vectors.

$$B = \begin{pmatrix} b_1, & b_2 \ldots & b_n \\ | & | \quad \ldots & | \end{pmatrix} \qquad (|b_j| = 1,\ j = 1 \ldots n).$$

In this equation B_{ik} are the adjunct of B and $B'_{ik} = B_{ki}$ are the adjunct of B'. Applying Cauchy-Binet's equations for the minus of B_{ik} and B'_{ki} for fixed values of k and variable values for i we have

$$\begin{aligned} \sum_{i=1}^{n} B_{ik}^2 = \sum_{i=1}^{n} B'_{ki}\,B_{ik} &= (B'\,B)_{kk} \\ &= G\,(b_1, \ldots b_{k-1}, b_{k+1}, \ldots b_n) \leqq \\ &\leqq |b_1| \ldots |b_{k-1}| \cdot |b_{k+1}| \ldots |b_n| = 1. \end{aligned}$$

(1) Kantorowitsch, L. W.: Funktionalanalysis und angewandte Mathematik (russ). Uspechi matem. Nauk, T III 6 (28) 1948.

In this equation G(...) is Gram's determinants formed by the given vectors, the upper value of which can be determined by the above [2].

Therefore

$$\|B^{-1}\| = \left\|\left(\frac{B_{i,k}}{\det B}\right)\right\| \leqq \frac{1}{|\det B|}\sqrt{\sum_{i,k=1}^{n} B_{i,k}^2} \leqq \frac{\sqrt{n}}{|\det B|} \tag{9}$$

After these preliminaries we are now in the position to determine under which conditions x_{n+1} approaches x*. Hence u(x*) = 0, we have

$$x^* = x^* - L_n u(x^*)$$

and considering (4)

$$\begin{aligned} x_{n+1} - x^* &= x_n - x^* - L_n [u(x_n) - u(x^*)] \\ &= [E - L_n u'(x^*)](x_n - x^*) \\ &\quad - L_n [u(x_n) - u(x^*) - u'(x^*)(x_n - x^*)], \end{aligned}$$

and also

$$\begin{aligned} &|(x - u_x)(*_2)_{,n} - (*x)\,n - (u_x)\,n| \cdot \|u_T\| + \\ &\quad |*x - u_x| \cdot \|(*x)_{,n}\, u_T - E\| \leqq |*x - 1 + u_x| \end{aligned} \tag{10}$$

Each point $\bar{x}n = \sigma x_n + (1 - \sigma)\, x^* (0 < \sigma < 1)$ together with $x_n \in K$ and $x^* \in K$ of the connecting line belong to K. Taking (8) and (7) into consideration we can write

$$|u(x_n) - u(x^*) - u'(x^*)(x_n - x^*)| \leqq \frac{1}{2} n \sqrt{n}\, \varkappa |x_n - x^*|^2. \tag{11}$$

(2) Gantmacher: Matrizenrechnung, V. 1, P. 8 and 233. Berlin: Verl. d. Wiss. 1958

By expansion with the diagonal Matrix

$$\left(\ddots \frac{1}{|x_j - x_n|} \ddots\right)$$

we obtain

$$L_n = \left(\frac{x_0 - x_n}{|x_0 - x_n|} \cdots \frac{x_{n-1} - x_n}{|x_{n-1} - x_n|}\right)\left(\frac{u_0 - u_n}{|x_0 - x_n|} \cdots \cdots \frac{u_{n-1} - u_n}{|x_{n-1} - x_n|}\right)^{-1}$$
$$:= X \cdot U^{-1}. \tag{12}$$

$$g_j = \frac{u_j - u_n - u'(x^*)(x_j - x_n)}{|x_j - x_n|}$$
$$= \frac{u_j - u_n - u'(x_n)(x_j - x_n) + [u'(x_n) - u'(x^*)](x_j - x_n)}{|x_j - x_n|}$$

Then according to (8) and (7) we have

$$|g_j| \leq \frac{|u_j - u_n - u(x_n)(x_j - x_n)|}{|x_j - x_n|} + \|u'(x_n) - u'(x^*)\|$$
$$\leq \frac{1}{2} n \sqrt{n}\, \varkappa |x_j - x_n| + n \sqrt{n}\, \varkappa |x_n - x^*|$$
$$\leq 2 n \sqrt{n}\, \varkappa \cdot \max_{j=0}^{n} |x_j - x^*|,$$

because

$$|x_j - x_n| \leq |x_j - x^*| + |x_n - x^*|.$$

Therefore follows

$$\frac{u_j - u_n}{|x_j - x_n|} = u'(x^*) \frac{x_j - x_n}{|x_j - x_n|} + g_j$$
$$U = u'(x^*) X + (g_0, g_1 \cdots g_{n-1})$$
$$:= u'(x^*) X + G, \tag{13}$$

where

$$\|G\| \leq \sqrt{\sum_{j=0}^{n-1} g_j^2} \leq 2 n^2 \varkappa \max_{j=0}^{n} |x_j - x^*|. \tag{14}$$

The value of the determinate describing X depends only on the present position of the x_j, but not on the distance of the x_j with respect to x^*. Let us assume for the time being that the positions of the

x_j (j = 0, 1...n) are of a general nature, and also that the determinate for X does not equal 0. A somewhat more restrictive condition is elaborated below. Instead of (13) we may use the equivalent relationship

$$U = \{E + G \cdot X^{-1} \cdot [u'(x^*)]^{-1}\}\, u'(x^*)\, X, \tag{13'}$$

from this expression we may see that U is certainly inverse if

$$\|G X^{-1} [u'(x^*)]^{-1}\| \leqq \|G\| \cdot \|X^{-1}\| \cdot \|[u'(x^*)]^{-1}\| < 1$$

Since X is a Matrix composed of unit vectors, and if we take (9) and (14) into consideration this condition now takes the form

$$2 n^2 \varkappa \max_{j=0}^{n} |x_j - x^*| \cdot \frac{\sqrt{n}}{|\det X|} \cdot \|[u'(x^*)]^{-1}\| < 1. \tag{15}$$

From this we see that the x_j (j = 0, 1...n) satisfies the inequalities (15). We may further write:

$$L_n = X \cdot U^{-1} = [u'(x^*)]^{-1} \{E + G \cdot X^{-1} [u'(x^*)]^{-1}\}^{-1},$$

From the above expression can be seen that L_n represents an approximation for the inverse functional Matrix $[u'(x^*)]^{-1}$.

Furthermore [3]

$$\|L_n\| \leqq \frac{\|[u'(x^*)]^{-1}\|}{1 - \|G \cdot X^{-1} [u'(x^*)]^{-1}\|},$$

(3) Ljusternik-Sobolew: Elemente der Funktionalanalysis, P. 102, Berlin: Akademie-Verl. 1955.

and

$$\|L_n\| \leqq \frac{\|[u'(x^*)]^{-1}\|}{1-\frac{2\cdot n^{\frac{5}{2}}\varkappa\,\|[u'(x^*)]^{-1}\|}{|\det X|}\max\limits_{j=0}^{n}|x_j-x^*|} \tag{16}$$

and

$$\begin{aligned} E-L_n u'(x^*) &= [u'(x^*)]^{-1}\left\{E-\left(E+G\,X^{-1}\,[u'(x^*)]^{-1}\right)^{-1}\right\}u'(x^*) \\ &= [u'(x^*)]^{-1}\left\{E+G\cdot X^{-1}\cdot[u'(x^*)]^{-1}\right\}^{-1}\cdot\left\{\left(E+G\cdot X^{-1}\cdot[u'(x^*)]^{-1}\right)-E\right\}u'(x^*) \\ &= [u'(x^*)]^{-1}\left\{E+G\cdot X^{-1}\cdot[u'(x^*)]^{-1}\right\}^{-1}G\cdot X^{-1}; \end{aligned}$$

therefore

$$\|E-L_n u'(x^*)\| \leqq \frac{2\,n^{\frac{5}{2}}\varkappa\,\|[u'(x^*)]^{-1}\|\max\limits_{j=0}^{n}|x_j-x^*|}{|\det X|-2\,n^{\frac{5}{2}}\varkappa\,\|[u'(x^*)]^{-1}\|\max\limits_{j=0}^{n}|x_j-x^*|}. \tag{17}$$

Considering (10) and the condition (15) it follows now from (11), (16) and (17)

$$|x_{n+1}-x^*| \leqq \frac{1}{2}\cdot\frac{n^{\frac{3}{2}}\cdot\varkappa\cdot\|[u'(x^*)]^{-1}\|\left(4\,n\max\limits_{j=0}^{n}|x_j-x^*|+|\det X|\cdot|x_n-x^*|\right)}{|\det X|-2\,n^{\frac{5}{2}}\cdot\varkappa\,\|[u'(x^*)]^{-1}\|\max\limits_{j=0}^{n}|x_j-x^*|}|x_n-x^*|. \tag{18}$$

We now may obtain a modification of rule (4') by repeating the above considerations nearly to the letter

$$|x_{k+1}-x^*| \leqq \frac{1}{2}\cdot\frac{n^{\frac{3}{2}}\cdot\varkappa\cdot\|[u'(x^*)]^{-1}\|\cdot\left(4\,n\cdot\max\limits_{j=0}^{n}|x_{j_k}-x^*|+|\det X_k|\cdot|x_k-x^*|\right)}{|\det X_k|-2\,n^{\frac{5}{2}}\cdot\varkappa\,\|[u'(x^*)]^{-1}\|\max\limits_{j=0}^{n}|x_{j_k}-x^*|}|x_k-x^*|, \tag{18'}$$

The above applies if the x_{jk} (j = 0, 1...n) will satisfy only the condition

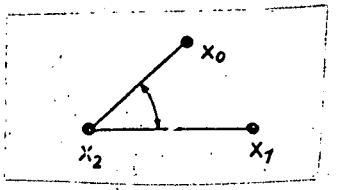
$$2n^2 \cdot \varkappa \max_{j=0}^{n} |x_{j_k} - x^*| \frac{\sqrt{n}}{|\det X_k|} \cdot \|[u'(x^*)]^{-1}\| < 1 \quad (15') \tag{15'}$$

with

$$X_k = \left(\frac{x_{0_k} - x_{n_k}}{|x_{0_k} - x_{n_k}|}, \ldots, \frac{x_{(n-1)_k} - x_{n_k}}{|x_{(n-1)_k} - x_{n_k}|} \right)$$

The determinants X respectively X_k are only a function of the relative positions of the x_j respectively x_{jk} to each other and may assume values between 0 (in case of a linear function) and 1 (in case they are orthogonal). In two or three dimensions the significance of the determinant X becomes very evident; for n = 2, we have

$$|\det X| = |\sin (\overrightarrow{x_2 x_0}, \overrightarrow{x_2 x_1})|$$



It is reasonable to demand that the determinant X does not become arbitrarily small. Let us therefore assume that ω is an arbitrary member between 0 and 1. From the beginning we shall consider in our calculations only such values of x_j and x_{jk} for which they "conditions of relative positions"

$$|\det X| \geqq \omega, \; |\det X_k| \geqq \omega \tag{19}$$

In order to make absolutely clear that conditions paragraph (15) is completely fulfilled we shall only consider such values of x_j which

are in the vicinity of x*. Thus

$$\max_{j=0}^{n} |x_j - x^*| < \frac{\omega}{2\, n^{\frac{5}{2}} \, \| u'(x^*)]^{-1} \| \varkappa} \tag{15*}$$

Thus we obtain

$$|x_{n+1} - x^*| \leqq \frac{1}{2} \cdot \frac{n^{\frac{3}{2}} \cdot \varkappa \cdot \| [u'(x^*)]^{-1} \| \, (4n+1) \max_{j=0}^{n} |x_j - x^*|}{\omega - 2\, n^{\frac{5}{2}} \cdot \varkappa \cdot \| [u'(x^*)]^{-1} \| \max_{j=0}^{n} |x_j - x^*|} \, |x_n - x^*|.$$

which is relatively weaker than (18). If the coefficients of $|x_n - x^*|$ in the last inequality is smaller than a given number q between 0 and 1, $x_n + 1$ is certainly an improvement over x_n. This is the case if

$$\max_{j=0}^{n} |x_j - x^*| \leqq \frac{2\, q\, \omega}{n^{\frac{3}{2}} \cdot \varkappa \cdot \| [u'(x^*)]^{-1} \| \, (4n + 4nq + 1)} := \varrho_1(\omega, q) \tag{20}$$

The existence of inequality (15*) is a consequence of the existence of the inequality (20). From the above we obtain

Rule 1: If the initial approximation $x_0, x_1 \ldots x_n \in K$ that for two numbers ω and q which lie between 0 and 1 the conditions

$$\left.\begin{array}{ll} \varepsilon) & |\det X| \geqq \omega \\ \zeta) & \max_{j=0}^{n} |x_j - x^*| \leqq \varrho_1(\omega, q) \quad \text{[siehe (20)]} \end{array}\right\} \tag{21}$$

is given, when the approximation x_{n+1} according to equation (4) approaches x* according to the inequality

$$|x_{n+1} - x^*| \leqq q\, |x_n - x^*|. \tag{22}$$

x_{n+2} can be now calculated from $x_1, x_2 \ldots x_n + 1$ if the condition ε is fulfilled, this means if

$$\left|\det\left(\frac{x_1 - x_{n+1}}{x_1 - x_{n+1}} \quad \cdots \quad \frac{x_n - x_{n+1}}{x_n - x_{n+1}}\right)\right| \geqq \omega$$

The values of x_{n+1} were previously calculated from $x_0 \ldots x_n$ in exactly the same manner. If this condition is violated, we may determine the next approximation x_{n+2} according to $x_{n+2} = x_{n+1} - L_n \cdot u(x_{n+1})$ by using the previously formed matrix factor n. In this case we have

$$|x_{n+2} - x^*| \leqq q\,|x_{n+1} - x^*|.$$

The same relationship also exists between the errors for the approximations x_{k+1} and the approximation x_k which were calculated according to (4'). This can be seen from the following: Because

$$\max_{j=0}^{n} |x_{j_k} - x^*| \leqq \max_{j=0}^{n} |x_j - x^*| \leqq \varrho_1(\omega, q) \qquad \text{[see (20)]}$$

$$|\det X_k| \geqq \omega \qquad \text{[see (19)]}$$

and

$$|x_k - x^*| \leqq \max_{j=0}^{n} |x_j - x^*| \leqq \varrho_1(\omega, q)$$

may be obtain from (18')

$$|x_{k+1} - x^*| \leqq \frac{1}{2} \cdot \frac{n^{\frac{3}{2}} \cdot \varkappa \cdot \left\|[u'(x^*)]^{-1}\right\| (4\,n\,\varrho_1 + \varrho_1)}{\omega - 2\,n^{\frac{5}{2}} \cdot \varkappa \cdot \left\|[u'(x^*)]^{-1}\right\| \cdot \varrho_1} |x_k - x^*|$$

$$\leqq q \cdot |x_k - x^*|.$$

The fact proven above may be expressed by the following rule.

Rule 2: In case x_{0k}, $x_{1k} \ldots x_{nk}$ are calculated approximations for which

$$\varepsilon) \quad |\det X_k| \geqq \omega \tag{23}$$

is valid, we can see that the approximation x_{k+1} which was calculated according to (4') approaches x* according to the inequality

$$|x_{k+1} - x^*| \leqq q \cdot |x_k - x^*| \leqq \ldots q^{k+1-n} \cdot |x_n - x^*|. \tag{24}$$

The series x_k approaches therefore x*.

If the conditions given by (21) are to be satisfied we always can take $L_k = L_n$ $(k > n)$ which means that $x_{jk} = x_j$ $(j = 0, 1 \ldots n)$.

The principle possibility of an approximate calculation of a null point of a non-linear system of equations with the aid of the generalized regula falsi is therefore sufficiently assured by the conditions $\varepsilon)$ and $\zeta)$. The appearance of a condition such as $\zeta)$ (that is "a sufficient" proximity relative to x*) was to be expected; on the other hand the condition of form $\varepsilon)$ initially seems to be only understandable if we consider it as a methodical aid. Other examples show that this is in some respects a required condition; for instance it might occur, that x_{n+1} received from x* in spite of arbitrary proximity of x_0, x_1 ...x_n in relation to x* (the x_{n+1} are not as good as the x_j) especially if these x_j are positioned sufficiently "badly" in relationship to each other that means, that the determinant X becomes very small.

For all practical purposes we should use as a rule only such values for x_{jk} for the construction of the convergent matrix factor

L_k for which the determinant X_k is greater than a given positive number, that is a number ω which is as close as possible to 1.

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